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Suppression of complex singularity using wedge interphase in interface fracture

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Abstract

A novel approach to suppress complex singularity associated with a bimaterial interface crack is proposed by including a wedge (made of a third material called interphase) with its vertex at the crack tip. The ensuing three-material wedge interface problem is analyzed using Williams' stress functions to extract the singularity. Different combinations of materials and wedge angles leading to real singularities are presented for some typical situations. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Interface crack; Singularity analysis; Wedge interphase model

1. Introduction

Research on an interfacial crack between two dissimilar materials was pioneered by Williams (1959), who used an eigenfunction approach to determine the singular character of the extensional stress near the tip of a crack at the interface between two materials. He found that the stresses have an oscillatory character with a maximum modulus determined by $r^{-1/2}$, where r is the distance from the crack tip. Later, Erdogan (1963), Sih and Rice (1964), England (1965), Erdogan (1965), Rice and Sih (1965) and Rice (1988) also confirmed the oscillatory nature of stresses at the tip of an interface crack. It is seen that the singularity is complex and is given by $1/2 + i\epsilon$ where ϵ is a bielastic constant. The imaginary part of this complex singularity (ϵ) gives rise to physically unrealistic stress oscillations ahead of the crack tip and periodic crack face interpenetration behind the tip. However, this behavior dominates only up to a certain distance from the crack tip after which these oscillations (and interpenetrations) die down.

A number of attempts were made to explain this seemingly nonphysical phenomenon. Rice (1988) and Shih and Asaro (1988, 1989) obtained conditions when the zone of oscillation was less than some physically relevant length. Others like Atkinson (1977), Comninou (1977a,b, 1978), Comninou and Schmueser (1979),

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Sinclair (1980), Ortiz and Blume (1990), Hills and Barber (1993) and Yang and Kim (1993) proposed alternative models of an interface crack. In this paper, eliminating the imaginary part of the singularity at an interface crack tip using a wedge made of a third material (interphase) forms the focus of this paper. This wedge interphase is located between the two material halves and ahead of the crack tip.

2. Earlier work

Atkinson (1977) attempted to provide an interface crack opening model free from overlapping by introducing a material sandwiched between the two materials comprising the original interface. The crack was assumed to lie wholly within the layer of additional material or at one of the two parallel interfaces. The elastic modulus of this layer varied continuously between the values of the original pair. Interpenetration was thus removed along with the original interface discontinuity.

Comninou (1977a,b, 1978) and Comninou and Schmueser (1979) precluded any crack face interpenetration by assuming crack faces to contact. The crack was considered to be closed near the tip with/without friction between the faces. Further, this contact zone was seen to be small in a tensile field and large when a shear field was present. While successful in removing interpenetration, the singular field in their model was antisymmetric about the crack and consequently crack propagation is restricted to mode II.

Sinclair (1980) presented a model for an interface crack tip free from interpenetration of crack flanks which permitted crack propagation in a crack opening mode. In this model, a finite COA was assumed such that the stresses had an inverse square-root singular nature necessary for a finite energy release rate. When the level of discontinuity was low, COAs were less than 60°; when the discontinuity was higher, larger COAs were predicted. He suggested that for the latter case, work of Comninou (1977a,b) was most appropriate. He concluded that for an interface crack, a degree of geometric nonlinearity was present and the deformed state had to be anticipated – either a region of contact or a crack opening angle, before loads were applied.

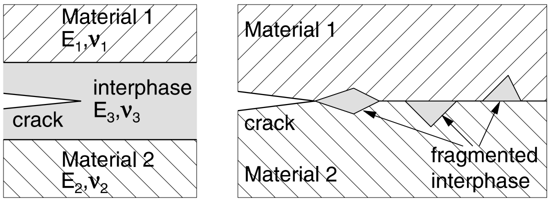
Ortiz and Blume (1990) considered interfacial tensile and shear strengths in their model; they allowed decohesion and sliding near an interface crack. They found that stress oscillations and crack face interpenetration were absent in this model.

Hills and Barber (1993) analyzed an interface crack assuming both open and closed crack tip cases. Open solutions were allowed only when contact zone was smaller than the process zone; however, they extended the range of the *open* formulation by embedding an appropriate *contact* field within the asymptotic crack tip field associated with the open solution. This approximation was tested against published solutions to various interface crack problems.

Yang and Kim (1993) proposed a new model of interface crack – with two shear-yield zones and one contact zone. They presented results of tensile, shear and combined (tensile and shear) stress fields. Effect of Dundurs' parameters α and β on shear-yield zone size and pressure distribution in contact zone was studied.

3. Wedge interphase model

The interphase model proposed by Atkinson (1977) had a third intervening layer between the two materials forming an interface as shown in Fig. 1(a). This layer was uniform and unbroken in the vicinity of the crack tip. In this paper, a different idea is proposed to model the interphase in order to suppress the complex singularity. If the interphase is discontinuous, an interface crack is likely to encounter fragments of interphase material. This fragmented interphase represents a series of inclusions along the interface (Fig. 1(b)). This interphase material may have the average properties of the straddling materials or may be an adhesive material. It will be shown that this configuration is capable of eliminating the



(a) (b)

Fig. 1. Continuous and fragmented interphase models.

complex singularity plaguing the analysis of interface cracks for a range of wedge angles and material properties. Recently, Mishuris (1997a,b, 1999) also speculated on a wedge interphase among other arbitrary shapes of interphase for a crack terminating perpendicular to the bimaterial interface. Different interfacial conditions near the crack tip were considered and emphasis was on mode III loading.

A series of papers by Hasebe and coworkers (Hasebe et al., 1989; Chen and Hasebe, 1996; Boniface and Hasebe, 1998), Ballarini (1990) and Asundi and Deng (1995) has examined the role of rigid elliptic and rhombic inclusions in both homogeneous and bimaterial problems. These investigations address the stress analysis aspects of crack initiation around inclusions. Here, the isolated rigid inclusion is replaced by an unbounded elastic wedge to set up the problem for a Williams analysis. Further, the emphasis here is on the evolution of singularity with changing geometry and elastic properties.

The *wedge interphase* model is shown in Fig. 2. In this three-material problem, the interphase in the form of a wedge ahead of the crack is defined by angles γ_1 and γ_2 corresponding to the two interfaces – marked as 1 and 2 in the figure. Material 1 has Young's modulus of E_1 and Poisson's ratio of ν_1. Corresponding values for material 2 are E_2 and ν_2, and for the interphase E_3 and ν_3.

Williams' stress function is used in this analysis. Williams (1952) gave a product solution as the Airy's stress function at a corner. This function has been successfully used at other regions of singularity like crack tips, etc. Extending the original function to include complex singularities,

$$\begin{aligned}\Phi_j(r,\theta) &= \mathrm{Re}\left\{r^{\lambda+1}\left[A_j\sin(\lambda+1)\theta + B_j\cos(\lambda+1)\theta + C_j\sin(\lambda-1)\theta + D_j\cos(\lambda-1)\theta\right]\right\} \\ &= \mathrm{Re}\left\{r^{\lambda+1}\hat{\Phi}_j(\lambda,\theta)\right\} \quad (j=1,2,3),\end{aligned} \tag{1}$$

where r and θ are polar coordinates of the point under consideration and λ, the stress singularity. A_j, B_j, C_j and D_j are complex constants for the three distinct regions – 1, 2 and 3 corresponding to material 1, 2 and interphase, respectively. Stress components in polar coordinates are given by

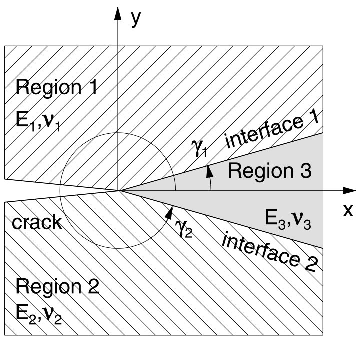


Fig. 2. Schematic diagram of wedge interphase model.

$$
\begin{aligned}
(\sigma_\theta)_j &= \mathrm{Re}\left\{r^{\lambda-1}\left[\lambda(\lambda+1)\ \hat{\Phi}_j(\lambda,\theta)\right]\right\} \\
&= \mathrm{Re}\left\{r^{\lambda-1}\lambda(\lambda+1)\left[A_j\sin(\lambda+1)\theta + B_j\cos(\lambda+1)\theta \right.\right.\\
&\qquad \left.\left. + C_j\sin(\lambda-1)\theta + D_j\cos(\lambda-1)\theta\right]\right\}, \\
(\sigma_r)_j &= \mathrm{Re}\left\{r^{\lambda-1}\left[\hat{\Phi}''_j(\lambda,\theta) + (\lambda+1)\ \hat{\Phi}_j(\lambda,\theta)\right]\right\} \\
&= -\mathrm{Re}\left\{r^{\lambda-1}\left[\lambda(\lambda+1)\left(A_j\sin(\lambda+1)\theta + B_j\cos(\lambda+1)\theta\right)\right.\right.\\
&\qquad \left.\left. + \lambda(\lambda-3)\left(C_j\sin(\lambda-1)\theta + D_j\cos(\lambda-1)\theta\right)\right]\right\}, \\
(\tau_{r\theta})_j &= -\mathrm{Re}\left\{r^{\lambda-1}\left[\lambda\ \hat{\Phi}'_j(\lambda,\theta)\right]\right\} \\
&= -\mathrm{Re}\left\{r^{\lambda-1}\lambda\left[(\lambda+1)\left(A_j\cos(\lambda+1)\theta - B_j\sin(\lambda+1)\theta\right)\right.\right.\\
&\qquad \left.\left. + (\lambda-1)\left(C_j\cos(\lambda-1)\theta - D_j\sin(\lambda-1)\theta\right)\right]\right\}.
\end{aligned}
\tag{2}
$$

The polar displacement components are

$$
\begin{aligned}
(u_r)_j &= \mathrm{Re}\left\{r^{\lambda}\frac{1}{2\mu_j}\left[-(\lambda+1)\ \hat{\Phi}_j(\lambda,\theta) + (1+\kappa_j)\left(C_j\sin(\lambda-1)\theta + D_j\cos(\lambda_j-1)\theta\right)\right]\right\} \\
&= \mathrm{Re}\left\{r^{\lambda}\frac{1}{2\mu_j}\left[-(\lambda+1)\left(A_j\sin(\lambda+1)\theta + B_j\cos(\lambda+1)\theta\right)\right.\right.\\
&\qquad \left.\left. -(\lambda-\kappa_j)\left(C_j\sin(\lambda-1)\theta + D_j\cos(\lambda-1)\theta\right)\right]\right\}, \\
(u_\theta)_j &= \mathrm{Re}\left\{r^{\lambda}\frac{1}{2\mu_j}\left[-\hat{\Phi}'_j(\lambda,\theta) - (1+\kappa_j)\left(C_j\cos(\lambda-1)\theta - D_j\sin(\lambda-1)\theta\right)\right]\right\} \\
&= \mathrm{Re}\left\{r^{\lambda}\frac{1}{2\mu_j}\left[-(\lambda+1)\left(A_j\cos(\lambda+1)\theta - B_j\sin(\lambda+1)\theta\right)\right.\right.\\
&\qquad \left.\left. -(\lambda+\kappa_j)\left(C_j\cos(\lambda-1)\theta - D_j\sin(\lambda-1)\theta\right)\right]\right\}.
\end{aligned}
\tag{3}
$$

Here $(\cdot)'$ denotes differentiation with respect to θ, and the Kolosov constant

$$
\kappa_j = \begin{cases} (3-4\nu), & \text{plane strain}, \\ (3-\nu)/(1+\nu), & \text{plane stress}. \end{cases}
\tag{4}
$$

This wedge interface problem has 12 boundary conditions corresponding to traction and displacement continuity across interfaces and traction-free crack faces. They are

1. traction-free crack faces:

$$
\begin{aligned}
(\sigma_\theta)_1 &= 0 \quad \text{along } \theta = \pi, \\
(\tau_{r\theta})_1 &= 0 \quad \text{along } \theta = \pi, \\
(\sigma_\theta)_2 &= 0 \quad \text{along } \theta = -\pi, \\
(\tau_{r\theta})_2 &= 0 \quad \text{along } \theta = -\pi;
\end{aligned}
$$

2. traction continuity across interface 1:

$$(\sigma_\theta)_1 = (\sigma_\theta)_3 \quad \text{along } \theta = \gamma_1,$$
$$(\tau_{r\theta})_1 = (\tau_{r\theta})_3 \quad \text{along } \theta = \gamma_1;$$

3. displacement continuity across interface 1:

$$(u_r)_1 = (u_r)_3 \quad \text{along } \theta = \gamma_1,$$
$$(u_\theta)_1 = (u_\theta)_3 \quad \text{along } \theta = \gamma_1;$$

4. traction continuity across interface 2:

$$(\sigma_\theta)_2 = (\sigma_\theta)_3 \quad \text{along } \theta = \gamma_2,$$
$$(\tau_{r\theta})_2 = (\tau_{r\theta})_3 \quad \text{along } \theta = \gamma_2;$$

5. displacement continuity across interface 2:

$$\begin{aligned}(u_r)_2 &= (u_r)_3 \quad \text{along } \theta = \gamma_2,\\ (u_\theta)_2 &= (u_\theta)_3 \quad \text{along } \theta = \gamma_2.\end{aligned} \tag{5}$$

Substituting expressions for stresses and displacements from Eqs. (2) and (3), the 12 boundary conditions are expressed in matrix form as

$$[\mathscr{S}]\mathscr{X} = 0, \tag{6}$$

where

$$\mathscr{X} = [A_1, B_1, \ldots, D_3]^{\mathrm{T}}.$$

For a nontrivial solution of the above homogeneous system of equations, determinant of the matrix $\mathscr{S}$ must vanish. Values of λ between 0 and 1 (leading to a singular stress field as seen from Eq. (2)), which result in a zero determinant are obtained numerically. It is observed that the *dominant* singularity is complex for small values of γ_1 and γ_2. This is expected as they relate closely to the interface crack geometry. On increasing these wedge angles to threshold values, the dominant singularity becomes real. These critical wedge angles depend on properties of the two materials and the interphase. This search for wedge angles is confined to a physically realistic range, i.e., $0 < \gamma_1 < \pi/2$ and $-\pi/2 < \gamma_2 < 0$. It may be noted that for some cases, *higher real* values of singularity are also seen. However, since they are neither dominant nor complex, they are not addressed further in this paper.

To illustrate this new model, specific geometries with particular material properties are chosen. Three classes of wedge geometry are studied: (a) symmetric about the crack line, (b) in the softer material and (c) in the harder material. Poisson's ratios of $v_1 = v_2 = 0.3$ are assumed and moduli ratio Γ $(= \mu_1/\mu_2)$ is varied from 2 to 11. Shear modulus of the interphase is taken as the average of the two materials straddling it. Further, three different Poisson's ratios are assumed for the interphase material (v_3) as follows.

(a) $v_3 = 1/2$

This relates to an interphase made of incompressible adhesive material. Wedge angles and corresponding dominant real singularities for this value of Poisson's ratio are shown in Figs. 3 and 4. It is seen that real singularities are possible in the two cases of a symmetric wedge and a wedge in the softer material. For a wedge in the stiffer material, however, real singularities result only for very small values of moduli ratio Γ – approximately between 1 and 1.5. For the symmetric case (Fig. 3), wedge angles range from 50° to 80° for Γ ranging from 2 to 11; corresponding dominant singularity variation is from 0.535 to 0.595. Wedge angles in the softer material (Fig. 4) vary from 38° to 58° in the same Γ range, while dominant singularity varies from 0.54 to 0.595.

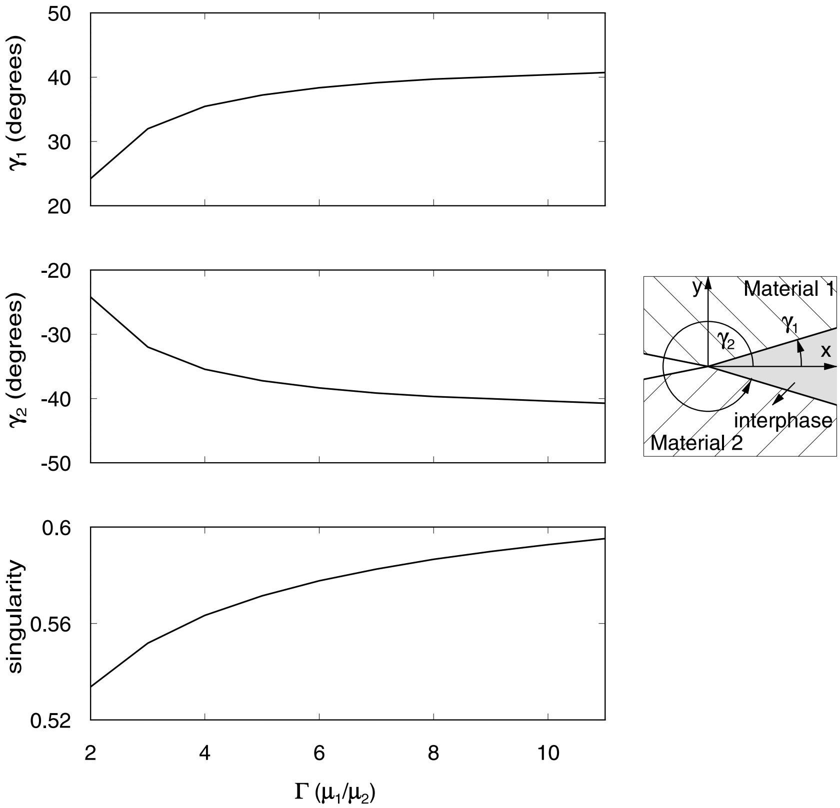


Fig. 3. Wedge angles and singularities for a soft interphase ($v_3 = 1/2$) located symmetrically, i.e., $\gamma_1 = -\gamma_2$.

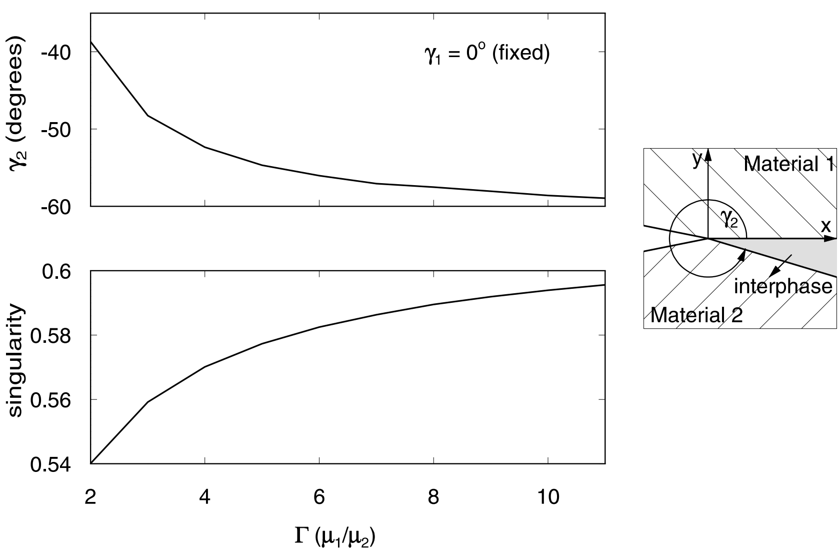


Fig. 4. Wedge angles and singularities for a soft interphase ($v_3 = 1/2$) located in the softer of the two material halves, i.e., $\gamma_1 = 0$.

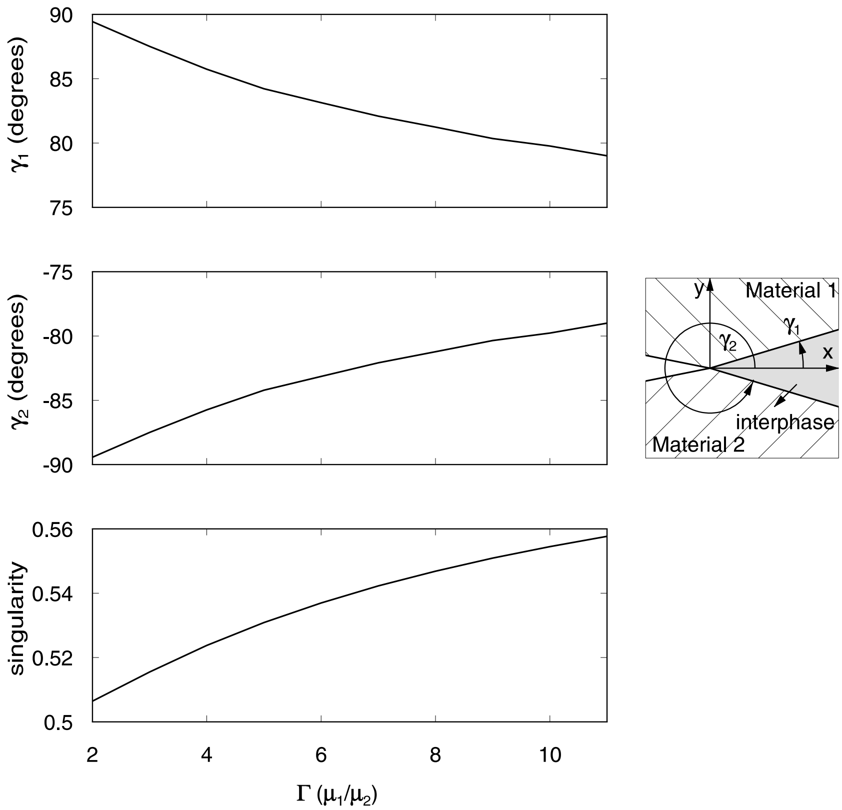


Fig. 5. Wedge angles and singularities for an interphase with $v_3 = (v_1 + v_2)/2$ located symmetrically, i.e., $\gamma_1 = -\gamma_2$.

(b) $v_3 = (v_1 + v_2)/2$

Fig. 5 shows wedge angles for this interphase with average properties of the two straddling materials for both shear modulus and Poisson's ratio. Here, real singularities result only for the symmetric case in the range of wedge angles under consideration. As before, corresponding values of the real dominant singularities are also shown. In this case, wedge angles vary from 176° to 156° for Γ ranging from 2 to 11 as in Fig. 5; corresponding dominant singularity variation is 0.51–0.555.

(c) $v_3 = 0$

In this case, it is seen that required wedge angles are outside the range considered, i.e., $\gamma_1 > \pi/2$ and/or $\gamma_2 < -\pi/2$ for all the three wedge geometries. Thus, these values are quite impractical and not reported here. It should be noted that these wedge angles are threshold values *above* which the singularity is real.

This analysis shows that the wedge interphase model is realistic only for cases (a) or (b), but not for case (c). Further, it is seen that threshold wedge angles for case (b) are very large compared to that for case (a). This implies that, for the material properties considered, the wedge interphase model is more appropriate for interphases exhibiting largely plastic behavior.

An interesting result is observed in case (b) – the threshold wedge angle becomes smaller for high Γ as seen in Fig. 5. For the earlier case (a), however (Figs. 3 and 4), the expected trend of increasing wedge angle with Γ is recovered.

It should be noted that the above conclusions correspond to the materials considered. These observations can be different for other sets of materials.

4. Discussion

Bonded joints necessarily bring in a third phase in the form of an adhesive. Even when two materials bond by diffusion, the transitional layer behaves differently from the bulk. Singularity at an interface crack in a bonded joint is a complicated function of the elastic and geometric make-up of the three-material junction. The magnitude of this singularity is critical to understand the failure and fracture of joints. It is seen that if the third material (interphase) is not considered, the resulting singularity is usually complex. Atkinson (1977) modeled this *interphase* as a thin layer of *uniform* thickness assuming the crack to bisect the layer or propagate along one of the two parallel interfaces. On the other hand, if this interphase is modeled as a wedge in the crack path, certain combinations of elastic constants and wedge angles yield real singularities. A significant outcome of the present study is that a three-material wedge weakens the singularity since λ is always more than 1/2. Thus, a wedge-like inclusion in the crack path may prove beneficial in adhesively bonded joints by reducing stress and strain gradients in the crack tip vicinity.

Sinclair (1980) nullified the imaginary part of singularity by considering crack opening, but restricted the singularity to what is manifest in a homogeneous medium, i.e., $\lambda = 1/2$. This restriction necessitated rather large angles of crack opening (not likely to be encountered in experiments) when Γ was high. However, if this restriction is relaxed by allowing real singularities *not necessarily equal to* 1/2 (Boniface and Simha, 1999), smaller crack opening angles are obtained.

Sinclair's COA model requires substantial changes in the *geometry behind* a crack tip – by allowing the crack flanks to move away from each other. On the other hand, the wedge interphase model strives to achieve similar results by modifying the *material ahead* of the crack tip – by including a wedge of interphase material. Consequently, Sinclair's model preserves the material combination, while the latter maintains the geometry. It is therefore clear that these two models offer different strategies to analyze interface cracks to achieve the same goal of real singularities. Thus, these two models, in a sense, complement each other.

Resolving the mathematical and conceptual differences proposed by different researches requires extremely refined experimental and computational models. In this respect earlier work by the authors (Boniface and Simha, 1999) has shown that stress fields predicted by different linear elastic interface crack models do not differ substantially to favor any one model over the other. Rather, all these models put together provide a better means of understanding the actual reality of the interface crack. In any case, the singular stress fields are always distorted by the nonlinear processes in the immediate interface crack tip vicinity. Experimental work in interface fracture has been intensely pursued following the development of new adhesives and advanced composites. Both static and dynamic experiments have been conducted to unravel stress and strain fields in the crack tip vicinity. Recently, Tippur and Xu (1995) and Xu and Tippur (1995) attempted to evaluate interface crack parameters using optical methods. Post-mortem examination revealed small fragments of foreign materials adhering to the fracture surfaces. In the case of bonded joints, randomly fragmented interphase material can be seen on the fractured surfaces. Some of these aspects can be addressed using the interphase wedge model presented in this paper.

5. Conclusions

The wedge interphase model is capable of suppressing the imaginary part of the interface crack singularity. This interphase ahead of the interface crack can be assumed to possess the average properties of the two material forming the interface. Other special interphases with $\nu = 0$ and $\nu = 1/2$ can also be considered. These cases are analyzed as a triple junction problem using Williams' approach and singularities are obtained. Combinations of wedge angle and material properties which eliminate imaginary part are determined. A salient point to note is that the wedge interphase model leads to a weaker singularity than what is manifest in a homogeneous medium. This aspect may have a beneficial effect on adhesively bonded joints.

The present triple junction wedge interphase model complements the work by Sinclair (1980) who allowed finite opening of the interface crack flanks.

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